# Polynomial Equation Approach to Control System Synthesis

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# Outline of the Presentation

**Basic tools:**
**parameterization of all stabilizing controllers**
**linear equations for polynomials (Diophantine equations)**

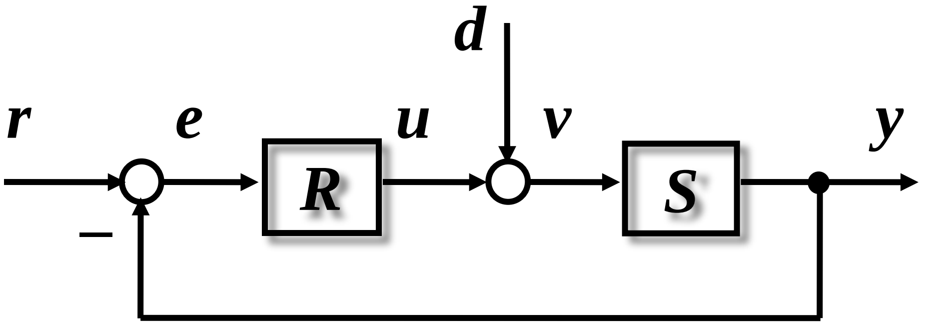
**Motivation, historical notes**

**Standard applications:**
**asymptotic properties, pole placement, deadbeat control,**
**$H_2$ optimal control, $l_1$ optimal control, robust control**

**Advanced applications:**
**stabilization subject to input constraints,**
**input and output shaping, fixed-order controller design**

# A Typical Control Problem

**Given a plant *S*,**
**determine a controller *R* so that**
**(1) the control system is stable, either Re$s$ < 0 or $|z|$ < 1**
**(2) additional specifications are met.**




**It is logical to stabilize first,**
**then meet the additional specifications.**

**Then one needs to determine *all* stabilizing controllers.**

# Polynomial Description

**Let $S = b/a$ and $R = q/p$ , coprime polynomial fractions.**

**Closed loop sensitivity**

$$H_S = \frac{1}{1+SR} = a\frac{p}{ap+bq} := aX$$

**and complementary sensitivity**

$$H_C = \frac{SR}{1+SR} = b\frac{q}{ap+bq} := bY$$

**In a stable system, $X$ and $Y$ are stable.**

**However, $X$ and $Y$ cannot be arbitrary since $H_S + H_C = 1$.**

**Hence**

$$aX + bY = 1$$

# Parameterization of Stabilizing Controllers

**All controllers that stabilize the plant $S = b/a$**
**are given by $R = Y/X$,**
**where $X, Y$ is a *stable rational* solution pair of**

$$aX + bY = 1$$

**All solution pairs can be expressed in parametric form as**

$$X = x + bW, \quad Y = y - aW$$

**where $x, y$ are polynomials such that $ax + by = 1$**
**and $W$ is a free *stable rational* parameter.**

**This is a fundamental result,**
**called the *Youla-Kučera parameterization*.**

# Example 1

**Plant**

$$S(s) = \frac{1}{s}$$

**Equation**

$$sx + y = 1$$

**A solution $x = 0$, $y = 1$ yields the stabilizing controllers**

$$R(s) = \frac{1 - sW}{W}, \quad W \neq 0 \text{ stable rational}$$

**For example,**
**$W = 1/(s+1)$ yields a proportional controller $R = 1$.**

**Taking $W = 1$ results in $R(s) = 1 - s$;**
**this controller is stabilizing but it is not proper**
**and the feedback system has a pole at $s = \infty$.**

# Dicsrete-Time Systems

**The parameterization applies to discrete-time systems as well.**

**Continuous-time systems can give rise to transfer functions that are not proper.**

**In the case of discrete-time systems, however, additional constraints have to be imposed: the transfer functions *S* and *R* are to be *proper* (so that the plant and the controller are causal systems) and one of them is to be *strictly proper* (so that the closed loop system is causal).**

**The chronology of samples in control systems is usually taken in such a way that *S* is strictly proper.**

# Example 2

**Plant**

$$S(z) = \frac{1}{z-1}$$

**Write**

$$S(z) = \frac{z^{-1}}{1-z^{-1}}$$

**Equation**

$$(1-z^{-1})x + z^{-1}y = 1$$

**A solution $x = 1$, $y = 1$ yields the stabilizing controllers**

$$R(z) = \frac{1-(1-z^{-1})W}{1+z^{-1}W}$$

**for any proper stable rational $W$.**

# Historical Notes

| | |
|---|---|
| Jury 1959 | deadbeat, SISO plant |
| Volgin 1962 | pole placement |
| Åström 1970 | minimum variance, minimum phase plant |
| Peterka 1972 | minimum variance |
| Kučera 1973 | stabilization, parameterization, SISO plant |
| | |
| Kučera 1975 | stabilization, parameterization |
| Youla et al 1976 | $H_2$ control, stabilization, parameterization |
| Kučera 1979 | polynomial equation approach |
| Desoer et al 1980 | proper stable fractions |
| Nett et al 1984 | state space formulas |

**It took decades to appreciate the importance of the result and come up with applications.**

# Additional Performance Specifications

- **There are as many stabilizing controllers for a given plant as stable rational free parameters *W*.**
- **The set of stabilizing controllers for a given plant contains controllers of arbitrarily high order.**
- **The parameter *W* in turn parameterizes all resulting stable closed-loop transfer functions and the parameterization is *linear* in *W*,**

$$\begin{bmatrix} v \\ y \end{bmatrix} = \frac{1}{1+SR}\begin{bmatrix} 1 & R \\ S & SR \end{bmatrix}\begin{bmatrix} d \\ r \end{bmatrix} = \begin{bmatrix} a(x+bW) & a(y-aW) \\ b(x+bW) & b(y-aW) \end{bmatrix}\begin{bmatrix} d \\ r \end{bmatrix}$$

**while it is *nonlinear* in *R*.**

# Asymptotic Properties

**Reference tracking:**
**output *y* follows reference *r* (error *e* goes to zero) asymptotically.**

**In terms of Laplace transforms,**
$e(s) = H_S(s)\,r(s)$ **is to be a stable rational function.**

**Disturbance attenuation:**
**effect of disturbance *d* on output *y* decreases asymptotically.**
**In terms of Laplace transforms,**
$y(s) = SH_S(s)\,d(s)$ **is to be a stable rational function.**

**This is to be achieved by a selection of the parameter *W*.**

# Example 3

**Plant**

$$S(s) = \frac{1}{s+1}$$

**Stabilizing controllers**

$$R(s) = \frac{1-(s+1)W}{W}$$

**for any stable rational $W \neq 0$.**
**Achievable sensitivity transfer functions are $H_S = (s + 1)W$.**

**To track a step reference, $r(s) = 1/s$, we have $e(s) = (s + 1)W/s$, so we must take $W = sW_1$ for any stable rational $W_1 \neq 0$.**
**To attenuate a sinusoidal disturbance, $d(s) = s/(s^2 + \omega^2)$, we constrain the parameter as $W = (s^2 + \omega^2)W_2$ for any stable rational $W_2 \neq 0$.**

**This demonstrates the *internal model principle*.**

# Pole Placement

**Plant** $S = b/a$

**Stabilizing controller** $R = Y/X$,

**where** $X = x + bW$, $Y = y - aW$ **and** $ax + by = 1$.

**Let** $W = w/d$, **where** $d$ **is a Hurwitz polynomial.**

**Then**

$$R = \frac{dy - aw}{dx + bw} := \frac{q}{p}$$

**Pole placement equation**

$$ap + bq = d(ax + by) = d$$

**The polynomial** $d$ **specifies the closed-loop poles while** $w$ **represents the remaining degrees of freedom.**

# Example 4

**Plant**

$$S(s) = \frac{1}{s-1}$$

**Stabilizing controllers**

$$R(s) = \frac{1-(s-1)W}{W}$$ **, $W \neq 0$ stable rational**

**Let the desired pole locations be given by $d(s) = s^2 + 2s + 1$.**

**Put $W = w/d$.**

**Then**

$$R(s) = \frac{(s^2+2s+1)-(s-1)w}{w}$$

**and for $R$ to have order 1, take $w(s) = s + \omega$ for any real $\omega$.**

**Otherwise poles at $s = \infty$ as well.**

# Deadbeat Control

**A *discrete-time* control problem.**

**Plant** $S = b/a$
**Find a stabilizing controller** $R = Y/X$
**such that all four closed- loop transfer functions**

$$H_S = a\,(x + bW), \quad SH_S = b(x + bW),$$
$$H_C = b(y - aW), \quad S^{-1}H_C = a(y - aW)$$

**are FIR (vanish in a finite/ shortest time).**
**This occurs iff** $W$ **is a *polynomial* in** $z^{-1}$**.**

**Special case of pole placement: all poles at** $z = 0$**.**
**Shortest transient time iff**
$x, y$ **is the least-degree solution pair of** $ax + by = 1$**.**

# Example 5

**Plant**

$$S(z) = \frac{z^{-1}}{1 - z^{-1}}$$

**Stabilizing controllers**

$$R(z) = \frac{1 - (1 - z^{-1})W}{1 + z^{-1}W}$$

**Then**

$$H_S = 1 - z^{-1} + z^{-1}(1 - z^{-1})W, \quad SH_S = z^{-1} + z^{-2}W,$$

$$H_C = z^{-1} - z^{-1}(1 - z^{-1})W, \quad S^{-1}H_C = 1 - z^{-1} - (1 - z^{-1})^2 W$$

**are all polynomials in $z^{-1}$ iff $W$ is a polynomial in $z^{-1}$.**

**The shortest impulse responses are achieved for $W = 0$.**
**The transients will vanish in one step.**

# $H_2$ Optimal Control

**Plant** $S = b/a$

**Find a stabilizing controller** $R = Y/X$

**such that, say,** $H_C = b(y - aW)$ **has a least** $H_2$ **norm.**

**Let** $\alpha\beta$ **be a polynomial**

**defined by keeping the stable (in Res < 0) zeros of** $ab$

**while replacing the unstable (in Res ≥ 0) ones**

**with their negative values.**

**In fact,** $\alpha$ **is the spectral factor of** $a(s)a(-s)$, $\beta$ **is that of** $b(s)b(-s)$.

**Then** $ab/\alpha\beta$ **is all-pass and**

$$\left\| H_C \right\|_2 = \left\| \frac{\alpha\beta}{ab} H_C \right\|_2 = \left\| \frac{\alpha y \beta}{a} - \alpha W \beta \right\|_2$$

# $H_2$ Optimal Control

**Consider the decomposition**

$$\frac{\alpha y \beta}{a} = r + \frac{q}{a}$$

**with $r$ polynomial and $q/a$ strictly proper.**

**With this decomposition,**

$$\left\| H_C \right\|_2^2 = \left\| \frac{q}{a} \right\|_2^2 + \left\| r - \alpha W \beta \right\|_2^2$$

**because $q/a$ and $r - \alpha W \beta$ are orthogonal**

**and thus the cross-terms contribute nothing to the norm.**

**The last expression is a complete square**

**whose first term is independent of $W$.**

**Hence the minimum is *unique* and achieved for $W = r/\alpha\beta$.**

# $H_2$ Optimal Control

**The $H_2$ optimal control is a special case of pole placement. Indeed,**

$$R = \frac{y - a\dfrac{r}{\alpha\beta}}{x + b\dfrac{r}{\alpha\beta}} = \frac{\alpha y\beta - ar}{\alpha x\beta + br} := \frac{q}{p}$$

**and**

$$ap + bq = a(\alpha x\beta + br) + b(\alpha y\beta - ar) = \alpha\beta(ax + by) = \alpha\beta$$

**The optimal closed-loop poles are given by $\alpha\beta$.**

**The pole placement equation has more than one solution. Which one is optimal? The one with $q/a$ strictly proper. It is the solution pair $p$, $q$ with $q$ having a least degree.**

# Example 6

**Plant**

$$S(s) = \frac{1}{s-1}$$

**Stabilizing controllers**

$$R(s) = \frac{1-(s-1)W}{W} \quad , W \neq 0 \text{ stable rational}$$

**The complementary sensitivity function to be minimized is**

$$H_C(s) = 1-(s-1)W$$

**Now $\alpha = s + 1$, $\beta = 1$**

**and the polynomial part of $\alpha y \beta / a = (s+1)/(s-1)$ is $r = 1$.**

**Thus $H_C$ attains minimum $H_2$ norm for**

$$W = \frac{1}{s+1}$$

**and the corresponding optimal controller is $R(s) = 2$.**

# Example 6

**Alternatively, one can solve the Diophantine equation**

$$(s-1)p+q=s+1$$

**for the solution pair $p$, $q$ such that $q/(s-1)$ is strictly proper. This yields the least-degree solution pair with respect to $q$, namely $p = 1$, $q = 2$.**

**The optimal controller is $R(s) = q/p = 2$.**

**In general, it is simpler to solve the polynomial equation than performing calculations with rational functions.**

# $l_1$ Optimal Control

**The $H_2$ norm minimization is appropriate for systems excited by finite energy signals.**

**When the exogenous signals persist, a more relevant norm to measure system performance is the $L_1$ norm (for continuous-time systems) or the $l_1$ norm (for discrete-time systems). The discrete-time case is much easier.**

**Plant $S = b/a$**

**Find a stabilizing controller $R = Y/X$**

**such that, say, $H_S = a(x + bW)$ has a least $l_1$ norm.**

# $l_1$ Optimal Control

**The optimal sensitivity function $H_S = a(x + bW)$ is not unique but it has a FIR property.**

**Perform stable-unstable factorizations $a = a^+a^-$ and $b = b^+b^-$, where $a^-$ and $b^-$ absorb all the zeros of $a$ and $b$, respectively, in the open unit disc $|z^{-1}| < 1$.**

**Then $H_S$ is a polynomial in $z^{-1}$ iff $W$ has the form**

$$W = \frac{w}{a^+b^+},$$

**where $w$ is a free polynomial.**

**Indeed, $H_S = ax + a^-b^-w$ and the $l_1$-norm minimization of $H_S$ is equivalent to a finite linear program for the coefficients of $w$.**

# Example 7

**Plant**

$$S(z) = z^{-1} \frac{z^{-1} - 1.5}{(1 - z^{-1})^2}$$

**Equation**

$$(1 - z^{-1})^2 x + z^{-1}(z^{-1} - 1.5) y = 1$$

**A solution** $x = 1 - 0.5z^{-1}$, $y = -3 + 2z^{-1}$

**yields the set of stabilizing controllers**

$$R(z) = \frac{-3 + 2z^{-1} - (1 - 2z^{-1})^2 W}{1 - 0.5z^{-1} + z^{-1}(z^{-1} - 1.5)W}$$

**for any proper and stable rational parameter *W*.**

# Example 7

**The set of achievable sensitivity functions is**

$$H_S(z^{-1}) = (1-2z^{-1})^2(1-0.5z^{-1}) + z^{-1}(1-2z^{-1})^2(z^{-1}-1.5)W$$

**and those which are *polynomials* in $z^{-1}$ are**

$$H_S(z^{-1}) = (1-2z^{-1})^2(1-0.5z^{-1}) + z^{-1}(1-2z^{-1})^2 w$$

**where $w$ is the numerator polynomial in $z^{-1}$ of**

$$W = \frac{w}{z^{-1}-1.5}$$

# Example 7

**An upper bound for the degree of $w$ is 2.**

**The linear program:**

**minimize** $t = r_1 + r_2 + r_3 + r_4 + r_5$

**subject to** $-r_i \le h_i \le r_i$ **and** $r_i \ge 0,\ i = 1, 2, \ldots, 5$

**where**

$$\begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \end{bmatrix} = \begin{bmatrix} -4.5 \\ 6 \\ -2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 4 & -4 & 1 \\ 0 & 4 & -4 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix}$$

**then returns** $w_0 = 1.5,\ w_1 = 0,\ w_2 = 0$ **so that**

$$W = \frac{1.5}{z^{-1} - 1.5}$$

# Example 7

**The optimal controller is**

$$R(z) = \frac{3-4z^{-1}}{(1+z^{-1})(z^{-1}-1.5)},$$

**the corresponding optimal sensitivity function is**

$$H_S(z) = 1-3z^{-1}+4z^{-2}.$$

**It is to be noted that *R* is *not* a deadbeat controller because $SH_S$ is not a polynomial.**
**Indeed, only polynomial parameters *W* result in deadbeat controllers.**

# $H_\infty$ Optimal Control

**The $H_\infty$ norm measures**
**the transfer of energy through the system.**

**Plant $S = b/a$**
**Find a stabilizing controller $R = Y/X$**
**such that, say, $SH_S = b(x + bW)$ has a least $H_\infty$ norm.**

**This is a disturbance attenuation problem**
**for $d \in L_2$ (continuous time) or $d \in l_2$ (discrete time).**

**The discrete-time case has a lot cleaner solution**
**than the continuous-time one.**

# $H_\infty$ Optimal Control

**The set of proper and stable real rational functions equipped with inner product**

$$\langle a,b\rangle = \frac{1}{2\pi j}\oint_C a(z^{-1})b(z)dz$$

**where C is the unit circle oriented counterclockwise, forms a linear space denoted by $H_2$.**

**The problem of minimizing**

$$\|SH_S\|_\infty = \|f - gh\|_\infty$$

**where $f := bx,\; g := -b^- b^-,\; h := b^+ b^+ W$ is to find the closest point to $f$ in the subspace $gH_2$, where the distances are measured using the $H_\infty$ norm.**

# $H_\infty$ Optimal Control

**Associated with each proper and stable rational function $F$ is a linear operator on $H_2$ denoted by $T_F$.**

**Let $N$ denote the orthogonal complement of $T_g$ in $H_2$ and let $\Pi$ denote the orthogonal projection mapping $H_2$ onto $N$.**

**Then $\Pi T_{f-gh}$ is the same for all $h$ and in fact equals $\Pi T_f$ .**
**Thus,**

$$\min_h \| f - gh \|_\infty = \| \Pi T_f \| .$$

**If $h_\infty$ attains this minimum**
**and $q \in N$ is a function such that $\| \Pi T_f q \|_2 = \| \Pi T_f \| \| q \|_2$**
**then the optimal sensitivity is**

$$( f - gh_\infty)(z) = (\Pi T_f q)(z) / q(z) .$$

# Example 8

**Plant**

$$S(z) = z^{-1}\frac{z^{-1}-2}{1-z^{-1}}$$

**Equation**

$$(1-z^{-1})x + z^{-1}(z^{-1}-2)y = 1$$

**A solution** $x = 1 - z^{-1}$, $y = 1$

**yields the set of stabilizing controllers**

$$R(z) = \frac{-1-(1-z^{-1})W}{1-z^{-1}+z^{-1}(z^{-1}-2)W}$$

**for any proper and stable rational parameter *W*.**

# Example 8

**The task is to minimize the $H_\infty$ norm of**

$$SH_S(z) = z^{-1}(z^{-1} - 2)(1 - z^{-1}) + z^{-2}(z^{-1} - 2)^2 W$$

**Thus**

$$f := z^{-1}(z^{-1} - 2)(1 - z^{-1}), \;\; g := -z^{-2}, \;\; h := (z^{-1} - 2)^2 W$$

**Since $g$ has two zeros, the subspace $N$ has dimension 2 and an orthonormal basis for $N$ is given by 1, $z^{-1}$.**
**The matrix representation $A$ of $\Pi T_f$ with respect to this basis is**

$$A = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix}$$

# Example 8

**Now $\|\Pi T_f\| = \bar{\sigma}(A)$, the largest singular value of *A*,**
**and $q(z)$, $(\Pi T_f q)(z)$ are respectively given,**
**with respect to the basis of *N*, by *p*, *Ap*,**
**where *p* is the eigenvector of $A^T A$ corresponding to $\bar{\sigma}^2(A)$.**

**In this case, $\bar{\sigma}(A) = 2$ and $q(z) = 1$, $(\Pi T_f q)(z) = -2z^{-1}$.**
**It follows that the optimal disturbance-to-output transfer function is**

$$SH_S(z) = -2z^{-1}$$

**the optimal parameter**

$$W_\infty(z) = \frac{0.5z^{-1} - 1.75}{(z^{-1} - 1.5)^2}$$

**and the corresponding optimal controller**

$$R_\infty(z) = \frac{0.5}{z^{-1} - 2}$$

# Robust Stabilization

**The notion of robust stability**
**addresses stabilization of plants subject to modeling errors,**
**when the actual plant may differ from the nominal model,**
**using a *fixed* controller.**

**The ultimate goal is to stabilize the actual plant.**
**The actual plant is unknown, however,**
**so the best one can do is to stabilize**
**a large enough set of plants.**

**The set of plants is constructed**
**as a neighborhood of the nominal plant.**
**The size of the neighborhood is measured by a suitable norm,**
**most common being the $H_\infty$ norm.**

# Model of Uncertainty

**Consider a nominal plant with transfer function $S$**
**and its neighborhood $S_\Delta$ defined by $S_\Delta := (1 + \Delta F)S$,**
**where $F$ is a *fixed* stable rational function**
**and $\Delta$ is a *variable* stable rational function such that $\|\Delta\|_\infty \le 1$.**

**Note that $\Delta F$ is the normalized plant perturbation away from 1**

$$\frac{S_\Delta}{S} - 1 = \Delta F$$

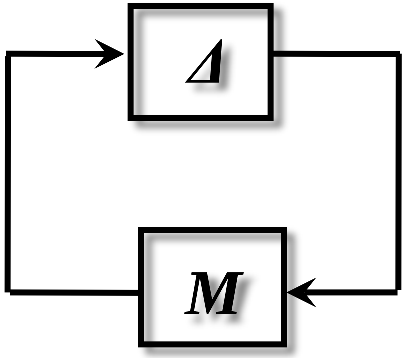
**Hence if $\|\Delta\|_\infty \le 1$, then for all frequencies $\omega$**

$$\left| \frac{S_\Delta(j\omega)}{S(j\omega} - 1 \right| \le |F(j\omega)|$$

**so $|F(j\omega)|$ provides the uncertainty profile**
**while $\Delta$ accounts for phase uncertainty.**

# Small Gain Theorem

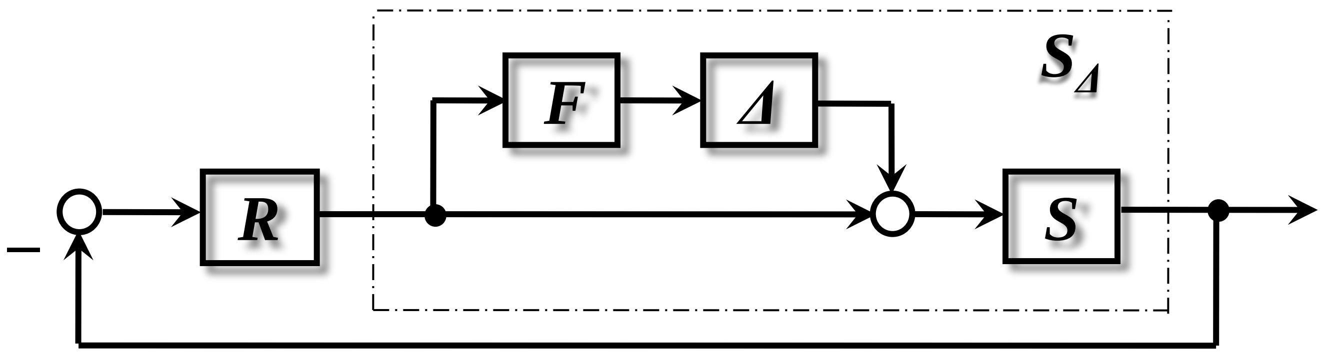
**Consider the *M-Δ* feedback system:**




**Suppose that $M$ is stable.**
**Then the feedback system is stable**
**for all stable $\Delta$ with $\|\Delta\|_\infty \le 1$**
**if and only if $\|M\|_\infty < 1$.**

# Robust Stability Condition

**The given model of uncertainty**




**collapses to an *M-Δ* feedback system with**

$$M = -F\,\frac{SR}{1+SR}$$

**Suppose that *R* stabilizes the nominal plant *S*.**
**Then *R* will stabilize the entire family of plants $S_\Delta$ iff**

$$\left\| F H_C \right\|_\infty < 1$$

# Parameterized Condition

**The set of all stabilizing controllers for *S* = *b/a***
**is described by the formula**

$$R = -\frac{y - aW}{x + bW}$$

**where *ax* + *by* = 1 and *W* is a free stable rational parameter.**
**The robust stability condition then reads**

$$\left\| Fb(y - aW) \right\|_\infty < 1$$

**Any stable rational *W* that satisfies this inequality**
**then defines a robustly stabilizing controller *R* for *S*.**

**In case *W* actually minimizes the norm**
**one obtains the best robustly stabilizing controller.**

# Example 9

**Plant**

$$S_\tau(s) = \frac{s+1}{s-1} e^{-\tau s}$$

**where the time delay $\tau$ is known only to the extent that it lies in the interval $0 \le \tau \le 0.2$ .**

**Find a controller that stabilizes the uncertain plant $S_\tau$.**

# Example 9

**The time-delay factor can be treated as a multiplicative perturbation of the nominal plant**

$$S(s) = \frac{s+1}{s-1}$$

**by embedding $S_\tau$ in the family $S_\Delta := (1 + \Delta F)S$, where $\Delta$ ranges over the set of stable rational functions such that $\|\Delta\|_\infty \le 1$.**

# Example 9

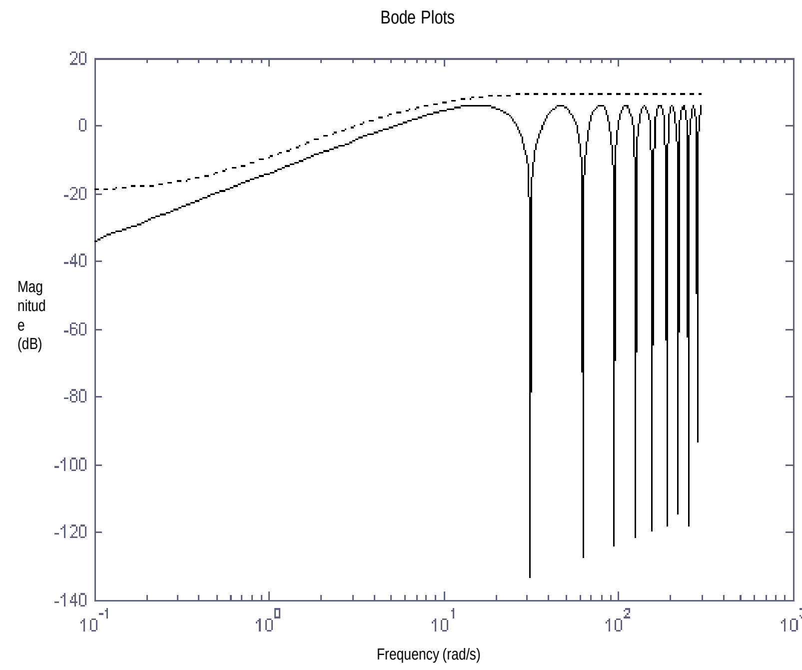
**To do this, *F* should be chosen so that**

$$\left|\frac{S_{\Delta}(j\omega)}{S(j\omega)}-1\right| = \left|e^{-j\omega\tau}-1\right| \leq \left|F(j\omega)\right|$$

**A suitable uncertainty profile is**

$$F(s)=\frac{3s+1}{s+9}$$

**Bode magnitude plot of this *F* and of $e^{j\omega\tau}-1$ for $\tau = 0.2$, the worst value**



# Example 9

**The set of all stabilizing controllers for the nominal plant *S* is**

$$R(s) = \frac{\frac{1}{2} - (s-1)W}{-\frac{1}{2} + (s+1)W}$$

**where $W \neq 1/2(s+1)$ is any stable rational parameter.**

# Example 9

**The robust stability condition reads**

$$\left\| P - QW \right\|_{\infty} < 1$$

**where**

$$P(s) = \tfrac{1}{2}(s+1)\frac{3s+1}{s+9}, \;\; Q(s) = (s-1)(s+1)\frac{3s+1}{s+9}$$

**The maximum modulus theorem implies**
**that the minimum of the $H_{\infty}$ norm**
**taken over all stable rational functions $W$ equals 0.4**
**and is achieved for**

$$W(s) = \frac{P(s) - P(1)}{Q(s)} = \frac{1}{10}\frac{15s+31}{(s+1)(3s+1)}$$

# Example 9

**Thus the robust stability condition is satisfied and the best robustly stabilizing controller is**

$$R(s) = \frac{2}{13}\frac{s+9}{s+1}$$

# Stabilization Subject to Input Constraints

**Most plants have inputs that are subject to hard limits on the range of variations that can be achieved.**

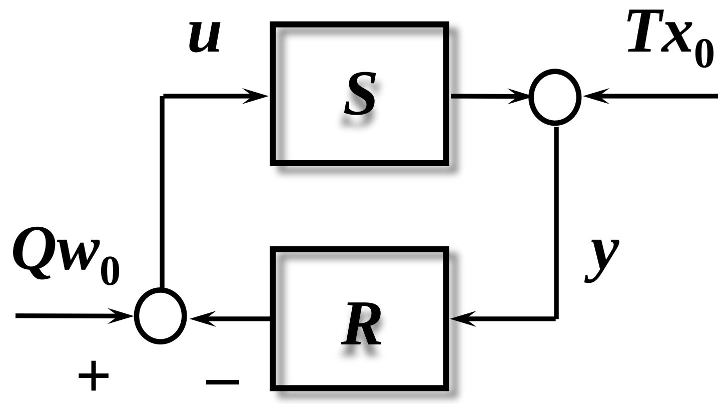
**Stabilization subject to input constraints:**

- **local stabilization,**
  **saturation prevented for a set of initial states,**
  **the control system behaves as a linear one**

- **global stabilization,**
  **saturation occurs, the control system is nonlinear**

# Problem Formulation

**Discrete-time control system**




**Find**

**a controller *R* such that**

**the control system is *locally asymptotically stable***

**for any initial state $x_0 \in P_F$**

$$P_F = \{\, x : Fx \le f \,\} \quad \text{polyhedron}$$

**and** $u(z) = u_0 + u_1 z^{-1} + u_2 z^{-2} + \dots$

$$-u^{-} \le u_k \le u^{+} \quad \text{constraint}$$

# Controller Parameterization

**Stabilizing controllers** $R = Y/X$

$$X = x + bW, \quad Y = y - aW$$

**Control sequence ($w_0 = 0$ assumed)**

$$u = -c\,(y - aW)\,x_0\,, \qquad W = p_0 + p_1 z^{-1} + \ldots$$

**is a linear function of the parameters $p_0, p_1, \ldots$ of the form**

$$u_k = G_k(p_0, p_1, \ldots),\ k = 0, 1, \ldots$$

**and it satisfies the constraint**
**if $x_0$ is in $P_G = \{\, x: G(p_0, p_1, \ldots)\, x \le g \,\}$**
**where**

$$G(p_0, p_1, \ldots) = \begin{bmatrix} G_0(p_0, p_1, \ldots) \\ -G_0(p_0, p_1, \ldots) \\ G_1(p_0, p_1, \ldots) \\ -G_1(p_0, p_1, \ldots) \\ \vdots \end{bmatrix}, \qquad g = \begin{bmatrix} u^+ \\ u^- \\ u^+ \\ u^- \\ \vdots \end{bmatrix}$$

# Polyhedron Inclusion

**Now $x_0$ is in $P_F$, so $P_F$ must be contained in $P_G$.**

***Farkas lemma*:**

**A polyhedron $P_F = \{ x\colon Fx \le f \}$**
**is contained**
**in a polyhedron $P_G = \{ x\colon Gx \le g \}$**
**if and only if there exists a matrix $P$**
**with non-negative entries**
**such that**

$$PF = G, \quad Pf \le g$$

# Solution

**The problem has a solution**
**if and only if there exist a matrix $P$ with non-negative entries and real numbers $p_0, p_1, \ldots$ such that**

$$PF = G(p_0, p_1, \ldots), \quad Pf \leq g$$

**This is a *linear program* for $P$ and $p_0, p_1, \ldots$**

**The stabilizing controller is then obtained by putting**

$$W = p_0 + p_1 z^{-1} + \ldots$$

**The program has a finite dimension**
**if $W$ is approximated by a polynomial.**

# Example 10

**Consider a plant described by input-output and state-output transfer functions of the form**

$$S(z) = \frac{z^{-1}}{1 - z^{-1}}, \quad T(z) = \frac{2}{1 - z^{-1}}$$

**The corresponding state equation**

$$x_{k+1} = x_k + 0.5u_k, \quad y_k = 2x_k$$

**The plant input is constrained as**

$$-1 \le u_k \le 1, \quad k = 0,1,...$$

**and the initial state $x_0$ belongs to the polyhedron**

$$P_F : \begin{bmatrix} 1 \\ -1 \end{bmatrix} x_0 \le \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \quad \text{(or } |x_0| \le 1/3\text{).}$$

# Example 10

**Stabilizing controllers**

$$R(z) = \frac{2-(1-2z^{-1})W}{1+z^{-1}W}$$

**for a free, proper stable rational parameter *W*.**
**The corresponding control sequence is**

$$u(z) = \left[-4 + 2(1-2z^{-1})W\right]x_0$$

**Now start with *W* = 0 and check whether the resulting linear program for *P* is feasible:**

$$P\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}-4\\4\end{bmatrix}, \quad P\begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\end{bmatrix} \leq \begin{bmatrix}1\\1\end{bmatrix}$$

**It is not, hence no controller of order 0 stabilizes the plant.**

# Example 10

**Proceed by choosing $W = p_0$ and check whether the resulting linear program for $p_0$ and $P$ is feasible:**

$$P\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}-4+2p_0\\4-2p_0\\-4p_0\\4p_0\end{bmatrix}, \quad P\begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\end{bmatrix} \le \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

**It is, and the solution**

$$p_0 = \frac{2}{3}, \quad P = \begin{bmatrix}0 & 8\\8 & 0\\0 & 8\\8 & 0\end{bmatrix}$$

**furnishes a stabilizing controller**

$$R(z) = \frac{4+4z^{-1}}{3+2z^{-1}}$$

# Example 10

**The actual polyhedron of stabilizable initial states is**

$$P_G:\ \frac{1}{3}\begin{bmatrix}-8\\8\\-8\\8\end{bmatrix}x_0 \le \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad \text{(or } |x_0| \le 3/8\text{)}$$

**and it includes $P_F$ as a proper subset.**

**Note that the closed-loop control system features the finite impulse response property.**

**Selecting a polynomial parameter $W$ implies that the closed-loop poles are all at the origin.**

# Input and Output Shaping

**Input constraints, but also output overshoot or undershoot**

**In discrete time, easy to handle.**
**The *z*-transform provides a simple direct relationship**

$$(y_0, y_1, y_2, \ldots) \leftrightarrow y_0 + y_1 z^{-1} + y_2 z^{-2} + \ldots$$

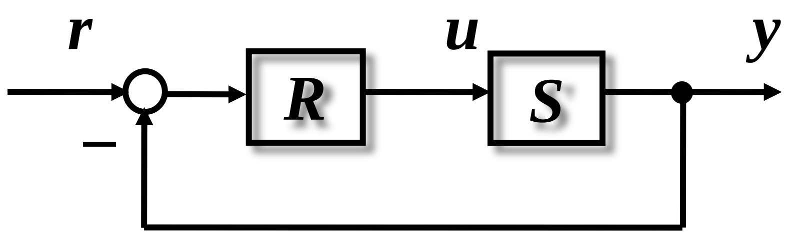
**Time domain constraints boil down to constraints on polynomial coefficients.**

**In continuous time, a new approach is needed:**

- **assign distinct negative real poles (rather than poles at $z = 0$)**
- **express time signals as polynomials in the corresponding exponential modes**

# Problem Formulation

**Given a plant $S = b/a$,**
**we are seeking a stabilizing controller $R = q/p$**
**such that the output $y$ asymptotically follows a reference $r$**




**while the time-domain constraints**
$u_{\min} \le u(t) \le u_{\max}$ , $y_{\min} \le y(t) \le y_{\max}$ **are satisfied for all $t \ge 0$,**
**where $u_{\min}$, $u_{\max}$, $y_{\min}$, and $y_{\max}$ are given real numbers.**

**We assume that $S$ is strictly proper**
**and that $R$ is proper so as to avoid impulsive modes.**

# Pole Assignment

**Assign distinct negative *integer* poles**

$$ap + bq = d \; := \prod_i (s - s_i)$$

**Then signals are sums of decaying exponentials modes**

$$u(t) = \sum_i u_i e^{-s_i t}, \quad y(t) = \sum_i y_i e^{-s_i t}$$

**Let $g$ be the greatest common divisor of the poles $s_i$**
**so that $s_i = k_i g$ for some integers $k_i$.**
**The signals can now be expressed as polynomials in $\lambda = e^{-gt}$**

$$u(\lambda) = \sum_i u_i \lambda^{k_i}, \quad y(\lambda) = \sum_i y_i \lambda^{k_i}$$

# Polynomial Non-Negativity Constraints

**When time $t$ increases from 0 to $\infty$,**

**indeterminate $\lambda$ decreases from 1 to 0**

**and the time constraints become the polynomial constraints**

$$u_{\min} \le u(\lambda) \le u_{\max}, \quad y_{\min} \le y(\lambda) \le y_{\max}$$

**or, equivalently, the polynomial non-negativity constraints**

$$u(\lambda) - u_{\min} \ge 0, \quad -u(\lambda) + u_{\max} \ge 0,$$
$$y(\lambda) - y_{\min} \ge 0, \quad -y(\lambda) + y_{\max} \ge 0,$$

**along the interval $\lambda \in [0, 1]$.**

# Convex LMI Constraint

**A polynomial non-negativity constraint**

$$p(\lambda) = \sum_{i=0}^{n} p_i \lambda^i \geq 0, \quad \text{for all } \lambda \in [\lambda_{\min}, \lambda_{\max}]$$

**is equivalent to the existence of real symmetric matrices $P_{\min}$, $P_{\max}$ of size $n + 1$ satisfying the linear matrix inequality constraints**

$$p_i = \text{trace}[P_{\min}(H_{i-1} - \lambda_{\min} H_i) + P_{\max}(\lambda_{\max} H_i - H_{i-1})], \;\; i = 0, 1, \ldots, n$$

$$P_{\min} \geq 0, \quad P_{\max} \geq 0$$

**where $H_i$ is the basis Hankel matrix with ones along the $(i + 1)$th anti-diagonal and zeros elsewhere.**

# Design Parameters

**Now all proper rational controllers *R***
**that assign the pole polynomial $d := \Pi_i\,(s - s_i)$**
**are parameterized by a numerator polynomial *w***
**of appropriate degree.**

**The coefficients of *w* are our design parameters**
**and they appear in the coefficients $u_i$ , $y_i$**
**of the closed-loop signals in an affine manner.**

**Therefore the linear matrix inequalities are convex**
**in the design parameters.**

# Example 11

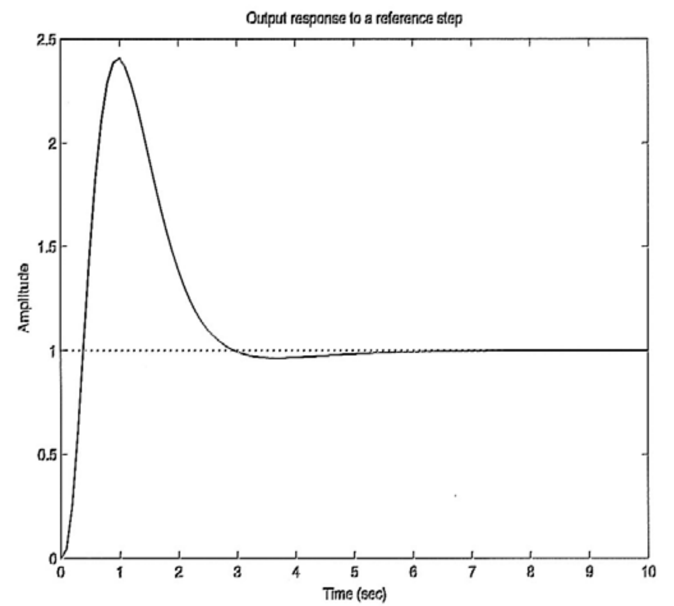
**Given the plant**

$$S(s) = \frac{s+0.5}{s(s-2)}$$

**the stabilizing controller**

$$R(s) = \frac{384s + 240}{s^3 + 17s^2 + 119s + 79}$$

**assigns the closed-loop poles at – 1, – 2, – 3, – 4, – 5 while ensuring asymptotic step reference tracking.**




**Despite the poles being negative real,**
**the step response features an unacceptable overshoot of 140 %**
**due to system zeros.**

# Example 11

**The set of all proper rational controllers that assign the above poles is given by**

$$R(s) = \frac{384s + 240 - s(s-2)w}{s^3 + 17s^2 + 119s + 79 + (s+0.5)w}$$

**where $w = w_0 + w_1 s$ is a free polynomial of degree at most 1.**

# Example 11

**The closed-loop responses to a step input are affine in *w*,**

$$y(s) = \frac{384s^2 + 432s + 120 - (s^3 - 1.5s^2 - s)w}{(s+1)(s+2)(s+3)(s+4)(s+5)}$$

**and correspond to a sum of decaying exponential modes in the time domain,**

$$y(t) = \sum_{i=0}^{5} y_i e^{-it}$$

**or to a polynomial**

$$y(\lambda) = \sum_{i=0}^{5} y_i \lambda^i$$

**in the indeterminate** $\lambda = e^{-t}$ **.**

**The coefficients** $y_i$ **are *linear* functions of** $w_0$ **and** $w_1$**.**

# Example 11

**Suppose the desired maximum overshoot is 20%**

$$y(t) \le 1.2\, y_0$$

**equivalent to the polynomial non-negativity constraint**

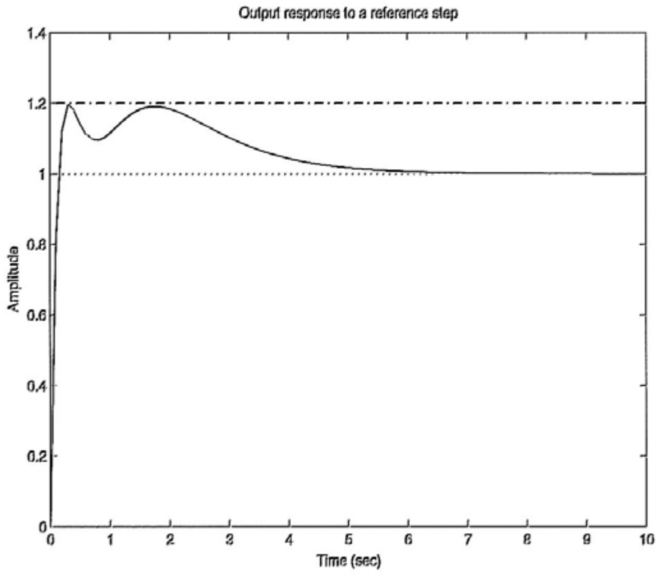
$$p(\lambda) = 1.2 y_0 - y(\lambda) = 0.2 y_0 - y_1\lambda - y_2\lambda^2 - y_3\lambda^3 - y_4\lambda^4 - y_5\lambda^5 \ge 0$$

**and in turn equivalent to an LMI in $w_0$ and $w_1$.**

**The LMI returns**

$$w(s) = -\,100.36 - 12.27s$$

**keeping the controller of order 3.**



# Fixed-Order Stabilizing Controllers

**A weakness of the sequential design**
**based on the Youla-Kučera parameterization**
**is that each performance specification beyond stability**
**may *increase the order* of the controller.**

**Actually, fixed-order stabilizing controllers**
**can be found by solving an LMI.**

# Polynomial Degree Control

**The degree control in the parameter $W = w/d$ is difficult.**

**If $d$ is fixed, all closed-loop transfer functions are affine in $w$ but the order of $w$ increases with each additional specification.**
**If $d$ is not fixed, we have a greater flexibility**
**but we run into difficulties as the set of stable polynomials is not convex in the space of coefficients.**

**The difficulty was resolved by providing a *convex inner approximation* of the non-convex stability domain in the space of polynomial coefficients.**
**This approximation is parameterized by a given polynomial, referred to as the *central* polynomial.**

# Problem Formulation

**Let us now show how to design stabilizing controllers of a fixed (presumably low) order.**

**Suppose a plant $S = b/a$ is given**
**and suppose that we have a stabilizing controller $\overline{R} = q/p$ .**

**We seek to find a stabilizing controller $R = y/x$**
**of a given order $m$, if such a controller exists.**

# The Two Controllers Relationship

**The two stabilizing controllers are related as**

$$p = x + bW, \quad q = y - aW, \quad \text{where } W = w/d.$$

**Then**

$$\underbrace{\begin{bmatrix} d & 0 & -p & b \\ 0 & d & -q & -a \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ d \\ w \end{bmatrix} = 0.$$

# Minimal Polynomial Basis

**Let**

$$\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \\ d_1 & d_2 \\ w_1 & w_2 \end{bmatrix}$$

**be a minimal polynomial basis of *A*.**
**Then all stabilizing controllers for *S* are**

$$R = (\lambda_1 y_1 + \lambda_2 y_2)/(\lambda_1 x_1 + \lambda_2 x_2)$$

**where $\lambda_1$ and $\lambda_2$ are polynomials**
**such that $\lambda_1 d_1 + \lambda_2 d_2$ is a *stable* polynomial.**

**A stabilizing controller of order *m* exists if**

$$\deg \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = m$$

**Alas, the set of *stable* polynomials is not convex.**

# Linear Matrix Inequality

**Given a fixed stable "central" polynomial *c*(*s*) of degree *n*, polynomial *d*(*s*) of degree *n* is stable if there exists a real symmetric matrix *Q* of size *n* solving the linear matrix inequality**

$$H_c(d,Q) = c^T d + d^T c - \varepsilon c^T c + \Pi_1^T Q \Pi_2 + \Pi_2^T Q \Pi_1 \geq 0$$

**where**

$$\Pi_1 = \begin{bmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \vdots \\ 0 & & & 1 \end{bmatrix}$$

**are projection matrices,**
***c* and *d* are the coefficient vectors of *c*(*s*) and *d*(*s*),**
**and ε is an arbitrarily small positive scalar.**

# Convex Inner Approximation

**The interpretation of this result is as follows:**
**as soon as polynomial *c* is fixed,**
**we obtain a sufficient linear matrix inequality condition**
**for stability of polynomial *d*.**

**Therefore,**

$$H_c = \{ d : \exists Q : H_c(d, Q) \geq 0 \}$$

**is a convex inner approximation of the (generally non-convex) stability domain in the space of polynomial coefficients around the central stable polynomial.**

# Problem Solution

**Using the convex inner approximation of the set of stable polynomials,**

**we can optimize over polynomials $\lambda_1$ and $\lambda_2$ to enforce low degrees of $x$ and $y$ (linear algebraic constraint)**

$$\deg \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = m$$

**as well as stability of $d$ (linear matrix inequality constraint)**

$$\begin{bmatrix} d_1 & d_2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = d$$

# Example 12

**Consider a plant of order 3,**

$$S(s) = \frac{1}{s(s^2 + s + 10)}.$$

**A stabilizing controller of order 2**
**can be found by placing the closed-loop poles at arbitrary locations. For example, the controller**

$$\overline{R}(s) = \frac{-26s^2 + 45s + 1}{s^2 + 4s - 4}$$

**places all five closed-loop poles at –1.**

**Find a lower order stabilizing controller.**

# Example 12

**A minimal polynomial basis for the polynomial matrix *A* is**

$$\begin{bmatrix} 0 & 1 \\ -1 & -26 \\ -1 & s^3+s^2+10s-26 \\ s^2+4s-4 & 149s-103 \end{bmatrix}$$

**All the stabilizing controllers**
**can be recovered from the polynomials $\lambda_1$ and $\lambda_2$**
**such that the pole polynomial**

$$d = -\lambda_1 + \lambda_2(s^3 + s^2 + 10s - 26)$$

**is stable.**

# Example 12

**From the first two rows of the basis a controller of order 0 can be obtained by restricting the parameters $\lambda_1$ and $\lambda_2$ to be constant.**

**Hurwitz stability criterion then reveals that $d$ is stable if and only if $\lambda_1 \in (-36, -26)$ and $\lambda_2 = 1$.**

**For example, with $\lambda_1 = -30$ we obtain the controller $R(s) = 4$ and the closed-loop pole polynomial $d(s) = s^3 + s^2 + 10s + 4$.**

**In this example, we were able to obtain an exact solution.
In general, the linear matrix inequality has to be used.**

# Summary

**The benefits of representing stabilizing controllers by a single parameter**

- **easy accommodation of additional design specifications by selecting an appropriate parameter**
- **all transfer functions in a stabilized system are *linear* in the parameter (while they are nonlinear in the controller)**
- **the parameter belongs to a smaller set of *stable* rational functions (while the controller is any rational)**

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