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FRACTIONAL DIGITAL CONTROL OF A HEAT SOLID: EXPERIMENTAL RESULTS

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Abstract: The purpose of this paper is to make a contribution for solving one of the major drawbacks for using fractional controllers: the controller implementation. In digital form, this is usually done by using a truncated version of the Grundwald-Letnikov formula for fractional derivative. In this work, experimental results are given comparing this method for digital implementation of a fractional PD$^{\delta}$ controller, with the alternative method resulting from the use of the trapezoidal rule and the continued fraction expansion for obtaining the discrete transfer function of the controller. From the results an interesting conclusion can be stated: the second method, which gives a controller in the form of a digital IIR filter, allows the use of lower order approximations of the fractional differential operator, that is, it reduces the memory and speed requirements of the digital system in which the controller is implemented.

Key words: Infinite Dimensional Systems, Fractional PID Control, Sampled Data Systems.

1 Introduction

The use of fractional calculus for modelling physical systems has been widely considered in the last decades [Mainardi 1996, Oldham et al. 1974]. We can also find works dealing with the application of this mathematical tool in control theory [Axtell et al. 1990, Dorčák 1994, Oustaloup 1995, Podlubny 1999a], but these works have usually theoretical character, whereas the number of works in which a real object is analyzed and a fractional-order controller is designed and implemented is small. The main reason for this fact, taking into account the theoretical advantages of fractional controllers for some control problems, is the difficulty of controller implementation. This difficulty arises from the mathematical nature of fractional operators, which, defined by convolution and implying a non-limited memory, demand hard requirements of processors memory and velocity capacities.

The purpose of this paper is to show how these requirements can be reduced without degradation in controlled system performances by using an alternative method proposed in [Vinagre et al. 2000]. This work uses the practical case described in [Petráš et al. 1998], where comparison between the use of a traditional PD controller and a fractional PD^δ controller to control the temperature of a heat solid (electric radiator) was made. Temperature is measured by a radiating pyrometer, filtered by an analogue active filter, and driven to host PC by a PCL 812 card. Control signal from analogue output on the PCL card is connected to the actuator (thyristor changer) where 0-5V signal is changed to 20-220V.

2 Fractional systems and controllers

2.1 Fundamental definitions in fractional calculus

Fractional calculus is a generalization of integration and differentiation to non-integer order fundamental operator ${}_aD_t^\alpha$, where α and t are the limits of the operation. The two definitions generally used for the fractional differintegral are the Grunwald-Letnikov (GL) definition and the Riemann-Liouville (RL) definition [Oldham et al. 1974]. The GL definition is:

$$ {}_aD_t^\alpha f(t) = \lim_{h\to 0}\frac{1}{h^\alpha}\sum_{j=0}^{\left[\frac{t-a}{h}\right]}(-1)^j\binom{\alpha}{j}f(t-jh) = \lim_{h\to 0}\frac{1}{h^\alpha}\Delta_h^\alpha f(t), \qquad (1) $$

where [.] means the integer part and where $\Delta_h^\alpha f(t)$ is the generalized finite difference of order α with step h. The RL definition is given by the expression

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-a)}\frac{d^n}{dt^n}\int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau, \qquad (2) $$

for (n - 1 < α < n) and where Γ(.) is the well known Euler's gamma function.

The Laplace transform method is used for solving engineering problems. The formula for the Laplace transform of the RL fractional derivative (2) has the form [Podlubny 1999b]:

$$ L\left\{{}_0D_t^\alpha f(t)\right\} = p^\alpha F(p) - \left[{}_0D_t^{\alpha-1} f(t)\right]_{t=0}, \qquad (3) $$

where p is a Laplace operator.

2.2 Fractional order systems

A fractional-order system can be represented by a fractional differential equation given by the following expression ($_0D_t^{\mu} \equiv D_t^{\mu}$):

$$a_n\, D_t^{\alpha_n} y(t) + \ldots + a_1\, D_t^{\alpha_1} y(t) + a_0\, D_t^{\alpha_0} y(t) = b_m\, D_t^{\beta_m} u(t) + \ldots + b_1\, D_t^{\beta_1} u(t) + b_0\, D_t^{\beta_0} u(t), \quad (4)$$

where α_k, β_k, (k = 0, 1, …, n) are generally real numbers and a_k, b_k, (k = 0, 1,…, n) are arbitrary constants.

For the obtaining of a discrete model for the fractional-order system (4), discrete approximations of the fractional-order operators have to be used. Then a general expression for the discrete transfer function of the system can be obtained in the form :

$$G\,(z) = \frac{b_m\left(\omega(z^{-1})\right)^{\beta_m} + \ldots + b_1\left(\omega(z^{-1})\right)^{\beta_1} + b_0\left(\omega(z^{-1})\right)^{\beta_0}}{a_n\left(\omega(z^{-1})\right)^{\alpha_n} + \ldots + a_1\left(\omega(z^{-1})\right)^{\alpha_1} + a_0\left(\omega(z^{-1})\right)^{\alpha_0}}, \quad (5)$$

where $\omega(z^{-1})$ denotes the discrete operator, expressed as a function of the complex variable z or the shift operator z^{-1}. In general, the discretization of fractional-order differentiator/integrator $p^{\pm r}$, ($r \in R$) can be expressed by the so-called generating function $p = \omega(z^{-1})$. This generating function and its expansion determine both the form of the approximation and the coefficients [Lubich 1986].

2.3 Fractional order controller

The fractional-order $PI^{\lambda}D^{\delta}$ controller can be described by the fractional-order differential equation [Podlubny 1999a]:

$$u(t) = Ke(t) + T_i\, D_t^{-\lambda} e(t) + T_d\, D_t^{\delta} e(t) \quad (6)$$

The discrete approximation of the fractional-order controller can be expressed as

$$C(z) = K + T_i\left(\omega(z^{-1})\right)^{-\lambda} + T_d\left(\omega(z^{-1})\right)^{\delta}, \quad (7)$$

where λ is an integral order, δ is a derivation order, K is a proportional constant, T_i is an integration constant and T_d is a derivation constant.

3 Digital realizations of fractional order controllers (FOC's)

The key point in digital implementation of a FOC is the discretization of the fractional order operators. In general, there are two discretization methods: *direct discretization* and *indirect discretization*. In *indirect discretization* methods two steps are required, i.e., frequency domain fitting in continuous time domain first and then discretizing the fit *p*-transfer function [Vinagre et al. 2000]. In this paper we compared two *direct discretization.*

3.1 Discretization using backward rule and power series expansion (PSE)

The simplest and most straightforward method is the direct discretization using finite memory length expansion from GL definition (1). This approach is based on the fact that, for a wide class of functions, the two definitions - GL and RL - are equivalent [Podlubny 1999b]. When backward difference rule is used, i.e., $\omega(z^{-1}) = (1 - z^{-1})/T$, performing the

PSE of $(1-z^{-1})^{\pm r}$ gives GL formula. By using the short memory principle [Podlubny 1999b], the discrete equivalent of the fractional-order integro-differential operator, $(\omega(z^{-1}))^{\pm r} \equiv D^{\pm r}(z)$, is given by

$$D^{\pm r}(z) = T^{\mp r} z^{-[L/T]} \sum_{j=0}^{[L/T]} (-1)^j \binom{\pm r}{j} z^{[L/T]-j} \tag{8}$$

where T is the sampling period, L is the memory length and the binomial coefficients are computed by expression [Dorčák 1994]:

$$c_0^{(r)} = 1, \qquad c_j^{(r)} = \left(1 - \frac{1+(\pm r)}{j}\right) c_{j-1}^{(r)} \tag{9}$$

It is very important to note that PSE scheme leads to approximations in the form of polynomials, that is, the discretized fractional order derivative is in the form of a FIR filter.

3.2 Discretization using trapezoidal rule and continued fraction expansion (CFE)

It is well known that, for interpolation or evaluation purposes, rational functions are sometimes superior to polynomials, roughly speaking, because of their ability to model functions with zeros and poles. In other words, for evaluation purposes, rational approximations frequently converge much more rapidly than PSE and have a wider domain of convergence in the complex plane.

So, for direct discretizing p^r, $(0 < r < 1)$, is of high interest the use of the trapezoidal rule or Tustin operator as generating function [Gorenflo 1996]. Furthermore, for control applications, the obtained approximate discrete-time rational transfer function should be stable and minimum phase. It can be shown that the proposed alternative discretization method, that is, the continued fraction expansion (CFE) of the Tustin rule, enjoy the above desirable properties. By using this method the discrete transfer function approximating fractional-order operators can be expressed as:

$$D^{\pm r}(z) = \left(\frac{2}{T}\right)^{\pm r} \mathrm{CFE}\left\{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^{\pm r}\right\}_{p,q} = \left(\frac{2}{T}\right)^{\pm r} \frac{P_p(z^{-1})}{Q_q(z^{-1})} \tag{10}$$

where T is the sample period and P and Q are polynomials of degrees p and q, respectively, in the variable z^{-1}.

4 Example: Temperature control of a solid

The mathematical model used for the system to be controlled is described by two-term transfer function in the form [Petráš et al. 1998]:

$$G(p) = \frac{1}{39.69 p^{1.26} + 0.598} \tag{11}$$

where parameters were obtained by an identification method based on minimization of the quadratics criteria - deference between measured values and model values.

The controller design was done according to the method described in [Petráš 1999] for stability measure $\sigma = 2.0$. The obtained fractional PD^{δ} controller designed for (11) has the continuous transfer function:

$$C(p) = 64.47 + 48.99 p^{0.5} \tag{12}$$

For implementing the controllers a position algorithm with reference digital prefiltering has been used. The transfer function of the digital prefilter is: $H(z) = 0.5/(1 - 0.5z^{-1})$. In experiments the following parameters have been used:

1. FIR : T=1 [s], L = 100 (order of the filter), 2. IIR: T=1 [s], p = q = 4 (order of the filter).

With these parameters the implemented controllers are:

$$C_1(z) = 64.47 + 48.99 \frac{\sum_{k=0}^{100} (-1)^k \binom{0.5}{k} z^{100-k}}{z^{100}} \tag{13}$$

$$C_2(z) = 64.47 + 48.99 \frac{0.3162z^4 - 1.0380z^3 + 1.2480z^2 - 0.6457z + 0.1195}{0.2564z^4 - 0.6395z^3 + 0.4887z^2 - 0.0779z - 0.0277} \tag{14}$$

Simulated step responses of the controlled system with controllers $C_1(z)$ and $C_2(z)$ are shown in Figure 1. In this figure it can be observed that the performances for both controllers are identical.

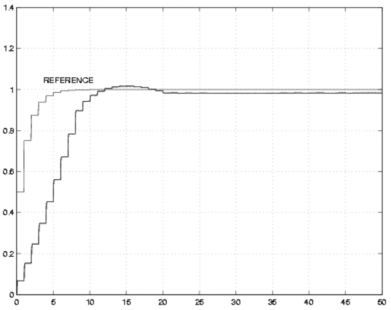

Figure 1. Simulated step responses

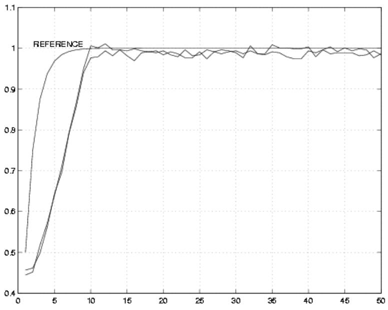

Figure 2. Measured step responses

Measured step responses of the controlled system with controllers $C_1(z)$ and $C_2(z)$ are shown in Figure 2. As in the case of simulations, the almost identical performances obtained with both controllers can be observed.

5 Conclusion

In this paper two of the alternative methods described in [Vinagre et al. 2000] have been used for digital realization of a fractional PD controller applied to temperature control of a solid (electric radiator). From the obtained results it can be concluded that for implementing the digital fractional controller is highly interesting to use Tustin rule and continued fraction expansion, because it reduces, without performance degradation, the digital system requirements (e.g. memory and computation time for control law implementation).

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