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A ROBUST STABILITY TEST PROCEDURE FOR A CLASS OF UNCERTAIN LTI FRACTIONAL ORDER SYSTEMS

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Abstract: For the first time, in this paper, a robust stability test procedure is proposed for linear time-invariant fractional order systems (LTI FOS) of commensurate orders with interval uncertain parameters. For the LTI FOS with no uncertainty, the existing stability test (or check) methods for dynamic systems with integer-orders such as Routh table technique, cannot be directly applied. This is due to the fact that the characteristic equation of the LTI FOS is, in general, not a polynomial but a pseudo-polynomial function of the fractional powers of the complex variable *s*. Of course, being the characteristic equation a function of a complex variable, stability test based on the argument principle can be applied. On the other hand, it has been shown, by several authors and by using several methods, that for the case of LTI FOS of commensurate order, a geometrical method based on the argument of the roots of the characteristic equation (a polynomial in this particular case) can be used for the stability check in the BIBO sense (bounded-input bounded-output). In this paper, we demonstrated this technique for the stability check for LTI FOS with parametric interval uncertainties through a worked-out illustrative example. In this example, time-domain analytical expression are available. Therefore, time-domain and frequency-domain stability test results can be cross-validated.

Key words: fractional order calculus, robust stability, parametric interval uncertainties, polynomial with fractional power, fractional order systems

1 Introduction

Stability is an asymptotic qualitative criterion of the quality of the control circuit and is the primary and necessary condition for the correct functioning of every control circuit. The existing methods developed so far for stability check are mainly for integer-order systems. However, for fractional order dynamic systems, it is difficult to evaluate the stability by simply examining its characteristic equation either by finding its dominant roots or by using other algebraic methods. At the moment, direct check of the stability of fractional order systems using polynomial criteria (e.g., Routh's or Jury's type) is not possible, because the characteristic equation of the system is, in general, not a polynomial but a pseudopolynomial function of fractional powers of the complex variable *s*. Thus there remain only geometrical methods of complex analysis based on the so called argument principle (e.g. Nyquist type) [Kemplfe 1996, Petráš et al. 1999].

When there are interval-type parametric uncertainties in the LTI FOS, to perform the robust stability test, we combine the above demonstrated stability test technique with the celebrated Kharitonov's Edge Theorem.

This contribution further clarifies the application of the indirect stability check method. We have also included a demonstrative example to verify the proposed robust stability test procedure.

2 Fractional order calculus

Fractional calculus is a generalization of integration and differentiation to non-integer order fundamental operator ${}_aD_t^\alpha$, where α and t are the limits of the operation. The two definitions generally used for the fractional differintegral are the Grunwald-Letnikov (GL) definition and the Riemann-Liouville (RL) definition [Oldham et al. 1974, Podlubny 1999]. The RL definition is given by the expression:

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(1-a)} \frac{d}{dt} \int_a^t \frac{f(\tau)}{(t-\tau)^\alpha} d\tau, \tag{1}$$

for $(0 < \alpha < 1)$ and where $\Gamma(.)$ is the well known Euler's gamma function. The formula for the Laplace transform of the RL fractional derivative (1) has the form [Podlubny 1999]:

$$ L\left\{{}_0D_t^\alpha f(t)\right\} = s^\alpha F(s) - \left[{}_0D_t^{\alpha-1} f(t)\right]_{t=0} \tag{2}$$

3 LTI fractional systems with commensurate order

A fractional order system can be represented by a transfer function of the form:

$$ G(s) = \frac{b_m s^{\mu_m} + \ldots + b_1 s^{\mu_1} + b_0 s^{\mu_0}}{a_n s^{\beta_n} + \ldots + a_1 s^{\beta_1} + a_0 s^{\beta_0}} = \frac{Q(s^{\mu_k})}{P(s^{\beta_k})}, \tag{3}$$

where μ_k, β_k, (k = 0, 1, 2, …) are generally real numbers $\beta_n > \dots \beta_1 > \beta_0$,, $\mu_m > \dots \mu_1 > \mu_0$, and a_k, b_k, (k = 0, 1, 2, …) are arbitrary constants. In the case of *commensurate order* systems it holds that, $\mu_i = \alpha k$, $\beta_i = \alpha k$, $k \in Z$, $\forall\ i$, and the transfer function has the form:

$$G(s) = K_0 \frac{\sum_{k=0}^{M} b_k s^{\alpha_k}}{\sum_{k=0}^{N} a_k s^{\alpha_k}} = K_0 \frac{Q(s^\alpha)}{P(s^\alpha)} \quad (4)$$

With $N > M$ the function $G(s)$ becomes a proper rational function in the complex variable s^α and can be expanded in partial fractions of the form:

$$G(s) = K_0 \left[\sum_{i=1}^{N} \frac{A_i}{s^\alpha + \lambda_i} \right] \quad (5)$$

where λ_i, i = 1,.., N are the roots of the polynomial $P(s^\alpha)$ or the system poles (assumed simple). The unit-step response of the system (5) can be expressed as

$$y(t) = L^{-1}\left\{ \frac{K_0}{s} \left[\sum_{i=1}^{N} \frac{A_i}{s^\alpha + \lambda_i} \right] \right\} = K_0 \sum_{i=1}^{N} A_i t^\alpha E_{\alpha,\alpha+1}(-\lambda_i t^\alpha), \quad (6)$$

being $E_{\nu,\gamma}(z)$ the Mittag-Leffler function in two parameters, which can be defined by using the gamma function $\Gamma(.)$ as [Mittag-Leffler 1904, Podlubny 1999]:

$$E_{\nu,\gamma}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\nu n + \gamma)} \quad (7)$$

4 Robust stability test procedure for LTI FOS

It is well-known that an integer order LTI system is stable if all the roots of the characteristic polynomial $P(s)$ are negative or have negative real parts if they are complex conjugate [Dorf et al. 1990]. This means that they are located on the left of the imaginary axis of the complex plane s. When dealing with commensurate order systems (or, in general, with fractional order systems) it is important to have in mind that $P(s^\alpha)$ is a multivalued function of s, the domain of which can be viewed as a Riemann surface [Pierre 1966]). Thus, we have to modify the stability statement as follows: *a commensurate order system is stable if its transfer function $G(s)$ is analytic in the right-half plane of the principal sheet of the Riemann surface and on the imaginary axis of said sheet.*

Since the principal sheet of the Riemann surface is defined by $-\pi < \arg(s) < \pi$, by using the mapping $\sigma = s^\alpha$, the corresponding σ domain is defined by $-\alpha\pi < \arg(\sigma) < \alpha\pi$, and the σ -plane region corresponding to the right-half plane of this sheet is defined by $-\alpha\,\pi/2 < \arg(\sigma) < \alpha\,\pi/2$.

Conjecture. Robust stability condition: The LTI FOS described by (4) with parametric interval uncertainties is robustly stable, if and only if four Kharitonov's polynomials of the equivalence system $H(\sigma)$ satisfy to the argument condition

$$|\arg(\sigma)| > \alpha\frac{\pi}{2}, \qquad \forall\sigma \in C, \qquad P(\sigma, q) = 0,$$

where transformation is: $s^\alpha \to \sigma$, for $0 < \alpha < 1$ and q is parametric uncertainty. When $\sigma = 0$ is a single root of P, the system cannot be stable.

The procedure for the robust stability check of the fractional order system can be divided to the following steps:

- **step1:** we rewrite the LTI FOS $G(s)$ of the commensurate order α, to the equivalence system $H(\sigma)$, where transformation is: $s^\alpha \to \sigma$, for $0 < \alpha < 1$;
- **step2:** we then write the interval polynomial $P(\sigma, q)$ of the equivalence system $H(\sigma)$, where interval polynomial is defined as $P(\sigma,q) = \sum_{i=0}^{n}[q^-, q^+]\sigma^i$;
- **step3:** for interval polynomial $P(\sigma,q)$, we have to write four Kharitonov's polynomials: $p^{--}(\sigma)$, $p^{-+}(\sigma)$, $p^{+-}(\sigma)$, $p^{++}(\sigma)$;
- **step4:** we test that four Kharitonov's polynomials satisfy to the stability condition: $|\arg(\sigma_i)| > \alpha\frac{\pi}{2}$, $\forall\sigma \in C$, being σ_i the i roots of $P(\sigma)$;

Simplification for low-degree polynomials: less Kharitonov's polynomials to be tested:

- degree 5: $p^{--}(\sigma)$, $p^{-+}(\sigma)$, $p^{+-}(\sigma)$
- degree 4: $p^{+-}(\sigma)$, $p^{++}(\sigma)$
- degree 3: $p^{+-}(\sigma)$

5 Illustrative example

We summarized a robust stability test procedure to answer the sample question given below. Given a family of the LTI FOS (with commensurate orders) described by

$$G(s,a,b) = \frac{1}{s^{1.5} + as^{0.5} + b} \tag{8}$$

where $a \in [0,1]$ and $b \in [1,2]$. Question: Is $G(s, a, b)$ robustly stable for all a and b ?

For the interval uncertainty of the parameters a and b we can show out the area of roots for characteristic equation of system (8) in the s-complex plane. As we can see in Figure 1, all roots are located in the left half plane.

Applying the robust stability theorem, however, we have the equivalence system

$$H(\sigma) = \frac{1}{\sigma^3 + a\sigma + b} \tag{9}$$

where the transformation is $s^{\alpha} \to \sigma$, $\alpha = 0.5$. The new characteristic polynomial with uncertain parameters a and b is $P(\sigma) = \sigma^3 + a\sigma + b = 0$. An interval polynomial then is

$$p(\sigma, q) = [1,2] + [0,1]\sigma + \sigma^3 \quad (10)$$

and four Kharitonov's polynomial are:

$$p^{--}(\sigma) = 1 + \sigma^3, \quad p^{-+}(\sigma) = 1 + \sigma + \sigma^3, \quad p^{+-}(\sigma) = 2 + \sigma^3, \quad p^{++}(\sigma) = 2 + \sigma + \sigma^3 \quad (11)$$

For our case, where polynomial degree is 3, will be sufficient to test the polynomial $p^{+-}(\sigma)$. The roots of the polynomial $p^{+-}(\sigma) = 2 + \sigma^3$ are: $\sigma_{1,2} = 0.629 \pm 1.091$, $\sigma_3 = -1.265$. Figure 2 shows the poles map, where all poles lie inside of the limited angular sector because $|\arg(\sigma)| > \pi/4$ for all σ, where $|\arg(\sigma_{1,2})| = 1.0472$ and $|\arg(\sigma_3)| = \pi$ and $\alpha = 0.5$.

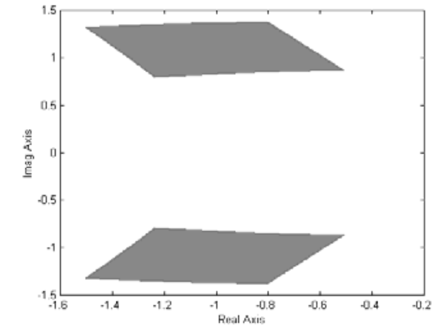

Figure 1. Poles map in s-complex plane for LTI FOS (8) with parametric interval uncertainties $a \in [0,1]$ and $b \in [0,1]$.

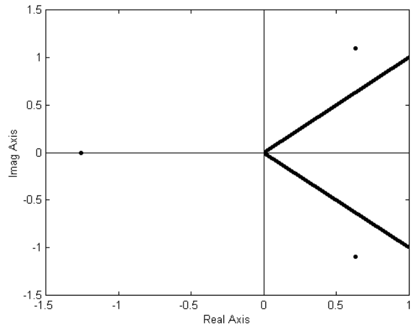

Figure 2. Poles map in σ - complex plane and stability limits (closed angular sector) for $\alpha = 0.5$.

The analytical solution (unit-step response) of LTI FOS (8) is based on the Mittag-Leffler function (7) and has the form [Podlubny 1999]:

$$y(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} b^k t^{1.5(k+1)1} E^{(k)}{}_{1,2.5+0.5k}(-at), \quad (12)$$

where $E^{(k)}_{\nu,\gamma}(\lambda t)$ is kth derivative of the Mittag-Leffler function in two parameters.

Figure 3 and Figure 4 show the simulation results for described (in legend) set of the parameters a and b.

The results given above which are based on the combination of Matignon's and Kharitonov's theorem and solution (12), depicted in Figure 3, for limits of the parameters a and b show that system $G(s, a, b)$ is *robustly stable* for all $a \in [0,1]$ and $b \in [0,1]$.

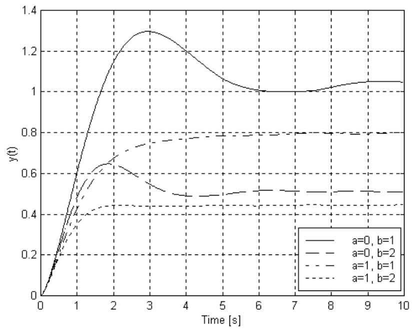

Figure 3. Unit-step responses for (8).

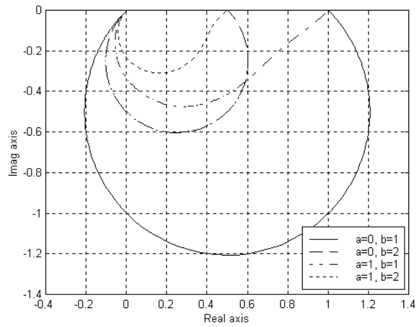

Figure 4. Nyquist plots for (8).

6 Conclusion

In this contribution we presented an illustrative example for checking the stability of LTI fractional order systems with commensurate orders and parametric interval uncertainties but parameters are known to lie within a known interval using the method given in Matignon's work [Matignon 1996] and Kharitonov's work (see e.g.[Henrion 2001]). The stability is checked also via classical methods given in [Dorf et al. 1990] and the obtained results are compared in time and frequency domain. It can be seen that the proposed check method is very simple and good for stability investigation of the LTI FOS.

We show only experimental approach to solution of the *robust stability* of the LTI FOS of commensurate orders with parametric interval uncertainties but clear and exact proof is still missing.

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